An example of obstructed surface

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School on deformation theory, September 16th - 20th

We work over $\mathbb{C}$. Consider

- $\mathbb{F}_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ del Pezzo surface in $\mathbb{P}^8$,
- $U := \mathcal{C}_a(\mathbb{F}_1, -K_{\mathbb{F}_1}) \subseteq \mathbb{A}^9$ affine cone,
- $X := \mathcal{C}_p(\mathbb{F}_1, -K_{\mathbb{F}_1}) \subseteq \mathbb{P}^9$ projective cone.

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If $x_1, \ldots, x_9$ are the homogeneous coordinates of $\mathbb{P}^8$, then $\mathbb{F}_1 \subset \mathbb{P}^8$ is the zero locus of

$$\begin{array}{cccc} x_5x_7 - x_2x_9 & x_6^2 - x_7x_9 & x_5x_6 - x_1x_9 & x_4x_6 - x_3x_9 \\ x_2x_6 - x_1x_6 & x_1x_6 - x_2x_9 & x_5^2 - x_4x_9 & x_4x_5 - x_8x_9 \\ x_2x_5 - x_4x_7 & x_1x_5 - x_3x_9 & x_4^2 - x_5x_8 & x_3x_4 - x_1x_8 \\ x_1x_4 - x_6x_8 & x_3^2 - x_2x_8 & x_1x_3 - x_7x_8 & x_1x_2 - x_3x_7 \\ x_3x_6 - x_4x_7 & x_3x_5 - x_6x_8 & x_2x_4 - x_7x_8 & x_1^2 - x_4x_7. \end{array} \tag{1}$$

If $x_1, \ldots, x_9$ are the affine coordinates of $\mathbb{A}^9$ and the last homogeneous coordinates of $\mathbb{P}^9$ and x_0 is the remaining homogeneous coordinate in $\mathbb{P}^9$, theese are also the zero locus for the embeddings of $U \subseteq \mathbb{A}^9$ and $X \subseteq \mathbb{P}^9$ respectively.

Theorem (Altmann, 1997)

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Idea of proof: Consider the map

$$\phi : \operatorname{Def}_X \to \operatorname{Def}_U$$

which restricts a deformation of X to U.
Using the spectral sequence for $\operatorname{Ext}^\bullet_X(\Omega_X, \mathcal{O}_X)$, one deduces that $E_2^{p,q} = H^q(X, \mathcal{E}xt^p_X(\Omega_X, \mathcal{O}_X)) = 0$ for all $q \geq 1$.
It follows that

$$\operatorname{Ext}^p_X(\Omega_X, \mathcal{O}_X) \to \operatorname{Ext}^p_U(\Omega_U, \mathcal{O}_U)$$

is an isomorphism for all $p \geq 1$. Since X is normal, $\operatorname{Ext}^1_X(\Omega_X, \mathcal{O}_X)$ and $\operatorname{Ext}^2_X(\Omega_X, \mathcal{O}_X)$ are respectively the tangent space and an obstruction space for Def_X.

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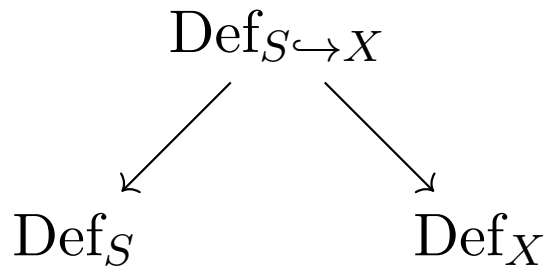
The natural transformations of functors



are smooth.

Idea of proof: The relative obstruction space for $\mathrm{Def}_{S\hookrightarrow X} \to \mathrm{Def}_X$ is $H^1(S, \mathcal{N}_{S/X}) = 0$. Consider the sequence $0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$ and tensor it with $\mathcal{O}_X(S)$. Observe that $\mathcal{N}_{S/X} = \mathcal{O}_X(S)_{|S}$,so that passing in cohomology it follows that $H^1(S, \mathcal{N}_{S/X}) = 0$.

The base of the miniversal deformation of S is $\operatorname{Spf} \frac{\mathbb{C}[[t,u_1,\ldots,u_m]]}{(t^2)}$, for some $m \geq 0$.

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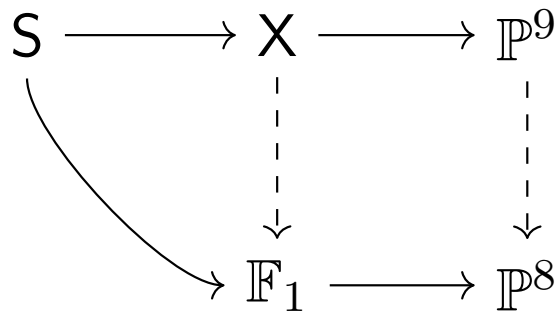
There is a finite and surjective morphism from S to $\mathbb{F}_1$:



Moreover the canonical bundle of S is very ample, so that S is a canonically polarized surface with everywhere non-reduced moduli space.

Thanks for your attention!